NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH, NATURAL RESOURCES AND APPLIED SCIENCES
SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

QUALIFICATION: Bachelor of Science in Applied Mathematics and Statistics	
QUALIFICATION CODE: 07BSAM	**LEVEL:** 6
COURSE CODE: PBT602S	**COURSE NAME:** Probability Theory 2
SESSION: JULY 2023	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

SUPPLEMENTARY / SECOND OPPORTUNITY EXAMINATION QUESTION PAPER	
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INSTRUCTIONS
1. There are 5 questions, answer ALL the questions by showing all the necessary steps.
2. Write clearly and neatly.
3. Number the answers clearly.
4. Round your answers to at least four decimal places, if applicable.

PERMISSIBLE MATERIALS

1. Nonprogrammable scientific calculators with no covers.

THIS QUESTION PAPER CONSISTS OF 3 PAGES (Including this front page)

Question 1 [14 marks]

1.1. Briefly explain the following:

1.1.1. Probability function [3]

1.1.2. Measure on sigma algebra $\sigma(S)$ [2]

1.1.3. Measure on a $\mathfrak{B}(S)$ algebra [2]

1.2. Show that if m and n are two measures on $\mathfrak{B}(S)$, then $m + n$ is measure on $\mathfrak{B}(S)$, where $(m+n)(A) = m(A) + n(A)$ [4]

1.3. Let $S = \{a, b, c\}$, then find:

1.3.1. Power set, $\mathcal{P}(S)$ [2]

1.3.2. Size of $\mathcal{P}(S)$ [1]

Question 2 [24 marks]

2.1. Let X be a continuous random variable with p.d.f. given by

$$f(x) = \begin{cases} x+1, & for\ -1 < x < 0, \\ 1-x, & for\ 0 \leq x < 1, \\ 0, & otherwise. \end{cases}$$

Then find cumulative density function of X. [7]

2.2. A hole is drilled in a sheet-metal component, and then a shaft is inserted through the hole. The shaft clearance is equal to the difference between the radius of the hole and the radius of the shaft. Let the random variable X denote the clearance, in millimeters. The probability density function of X is

$$f_X(x) = \begin{cases} C(1-x^4) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

2.2.1. Show that the value of $C = 1.25$. [3]

2.2.2. What is the probability that the clearance is between 0.3 mm and 0.8 mm? [4]

2.2.3. If $R = X + 0.3$, then find the expected value of R. [3]

2.3. The waiting time, in hours, between successive speeders spotted by a radar unit is a continuous random variable X with cumulative distribution function (c.d.f.)

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-8x} & \text{for } x \geq 0 \end{cases}$$

Then use the c.d.f. of X to find

2.3.1. $P(1 < X \leq 2)$ [2]

2.3.2. the median value of X [3]

2.3.3. the p.d.f. of X. [2]

Question 3 [18 marks]

3.1. Suppose that the joint p.d.f. of two continuous random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 12x, & 0 < y < x < 1;\ \ 0 < x^2 < y < 1, \\ 0, & elsewhere. \end{cases}$$

Find the marginal p.d.f. of Y. [3]

3.2. Let Y_1, Y_2, and Y_3 be three continuous random variables with the following joint p.d.f.

$$f(y_1, y_2, y_3) = \begin{cases} 6e^{-(y_1+2y_2+3y_3)}, & for\ y_i > 0;\ (i = 1,2,3), \\ 0, & elsewhere. \end{cases}$$

Then find

3.2.1. the marginal joint p.d.f of Y_1 and Y_3. **Hint:** just find $f(y_1, y_3)$. [4]

3.2.2. the conditional distribution of Y_2 given $Y_1 = 1, Y_3 = 1$. [3]

3.2.3. $P(Y_2 < 2|Y_1 = 1, Y_3 = 1)$. [3]

3.3. If X and Y are linearly related, in the sense that $Y = aX + b$, where $a > 0$, then show that $\rho_{XY} = 1$. **[5]**

QUESTION 4 [28 marks]

4.1. Let a random variable Z follows a standard normal distribution [i.e., $Z \sim N(0,1)$] with a p.d.f given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\ for - \infty < z < \infty$$

4.1.1. Show that the moment generating function of Z is given by $M_Z(t) = e^{\frac{1}{2}t^2}$. **[8]**

4.1.2. If $X \sim N(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$, then show that the moment generating function of X is $M_X(t) = e^{t\mu+\frac{1}{2}t^2\sigma^2}$. **Hint:** use the moment generating function of Z obtained above. **[6]**

4.1.3. Find the cumulant generating function of X and hence find the first cumulant. **[5]**

4.2. Let $X_1, X_2, \ldots. X_n$ be independently distributed with normal distribution with mean μ_k and variance σ_k^2, thus, $X_k \sim N(\mu_k, \sigma_k^2)$. If $Y = \sum_{i=1}^{n} X_i$,

4.2.1. Find the characteristics function of Y. **Hint:** $\phi_{X_k}(t) = e^{it\mu - \frac{t^2\sigma^2}{2}}$ **[7]**

4.2.2. Use the properties of characteristics function to comment on the distribution of Y. **[2]**

QUESTION 5 [26 marks]

5.1. Let Y be continuous random variable with a probability density function $f(y) > 0$. Also, let $U = h(Y)$. If h is increasing on the range of a given random variable, then show that [6]

$$f_U(u) = f_Y\left(h^{-1}(u)\right) \frac{d}{du} h^{-1}(u)$$

5.2. Let X_1 and X_2 be independent random variables with the joint probability density function given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & if\ x_1 > 0;\ x_2 > 0, \\ 0, & otherwise. \end{cases}$$

Find the joint probability density function of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$ [10]

=== END OF PAPER===

TOTAL MARKS: 100